Drag on an Oscillating Rod with Longitudinal and Torsional Motion

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An analytical expression for the viscous drag on a smooth circular cylindrical rodlike cable oscillating with longitudinal and torsional motion is obtained. Based on a linear damping law, an approximate value for the drag coefficient is also presented. The analysis and conclusions are useful when dealing with many ocean engineering problems of practical importance.

Introduction

FOR the analysis of many practical problems related to the sea-state excitation of long rods and cable-body systems for salvage, buoy, oil drilling, and towing operations, an expression for the viscous drag force acting on the longitudinal rod or cable is required. It has been shown that the inclusion of this term can have a pronounced effect on the tension in the rod or cable and subsequent response of the system. For example, the propagation and attenuation of internal stress waves and snap loads through the cable in a cable lifting system are affected by the external viscous damping.[1] The problem is also of considerable interest in offshore oil drilling problems since a similar analysis is usually performed.[2]

It is a common practice to postulate a linear relationship between the external damping force and the surface velocity of the rod when investigating the longitudinal vibrations based on the wave equation. The solution of this damped wave equation with an arbitrary damping coefficient can show the qualitative effects of external damping. However, the magnitude and frequency dependence of the damping coefficient can significantly alter the nature of the solution.

The purpose of this investigation is to formulate an analytical expression for the viscous drag on a smooth circular cylindrical rodlike cable oscillating with longitudinal and torsional motion. An examination of the hypothesis of the linear damping law will be made and the damping coefficient estimated.

Formulation of the Governing Equations

An infinitely long rod or cable with a circular cross section and no curvature is oscillating with longitudinal and torsional motion in an environment of fluid. Through the viscous effects at the interface of the solid and liquid, concentric waves are propagating outward from the cable exciting the fluid particles in an otherwise stationary state. The Navier-Stokes equations governing the motion of the viscous fluid can conveniently be expressed in a (r,θ,z) circular cylindrical coordinate system. After assuming that the flowfield is independent of θ and z, the equations become linear and are simplified to the form

$$v^2/r = (1/\rho)\partial p/\partial r \tag{1}$$

$$\frac{\partial v}{\partial t} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) \tag{2}$$

$$\frac{\partial w}{\partial t} = \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{3}$$

where $v(r,t)$ and $w(r,t)$ are the velocity components in the θ and z directions, respectively, p is the pressure, ρ the density, and ν the kinematic viscosity. It should be noted that the equation for v and w are uncoupled as a consequence of the assumption. The kinematic boundary conditions for the fluid particles at the surface of the circular solid of radius a which is oscillating with prescribed velocity $\tilde{V}_c = (V_0 \cos\omega t)\tilde{e}$ are at $r = a$

$$\tilde{V} = \tilde{V}_c$$

$$v = |\tilde{V}_c| \cos\beta = V_0 \cos\omega t \cos\beta$$

$$w = |\tilde{V}_c| \sin\beta = V_0 \cos\omega t \sin\beta$$

where

$$\tilde{e} = \cos\beta\tilde{e}_\theta + \sin\beta\tilde{e}_z$$

and as $r \to \infty$, v and w approach zero. If $\beta = 0°$ the rod has pure torsional oscillation and if $\beta = 90°$ it has pure longitudinal oscillation.

The tangential drag force $\tilde{F}$ acting on the rod per unit length is

$$\tilde{F} = -2\pi a(\tau_{r\theta}\tilde{e}_\theta + \tau_{rz}\tilde{e}_z) \tag{4}$$

or

$$\tilde{F} = -2\pi a\tau_w\tilde{\nu}$$

where

$$\tau_w = (\tau_{r\theta}{}^2 + \tau_{rz}{}^2)^{1/2} \tag{5}$$

$$\tau_{r\theta} = \mu(\partial v/\partial r - v/r)|_{r=a} \tag{6}$$

$$\tau_{rz} = \mu\partial w/\partial r|_{r=a} \tag{7}$$

and

$$\tilde{\nu} = \cos\gamma\tilde{e}_0 + \sin\gamma\tilde{e}_z$$

$$\tan\gamma = \tau_{rz}/\tau_{r\theta}$$

If the angle γ is equal to β, then the drag force is in an opposite direction to the velocity vector. However, this is yet to be established. The work done by this force on the fluid and

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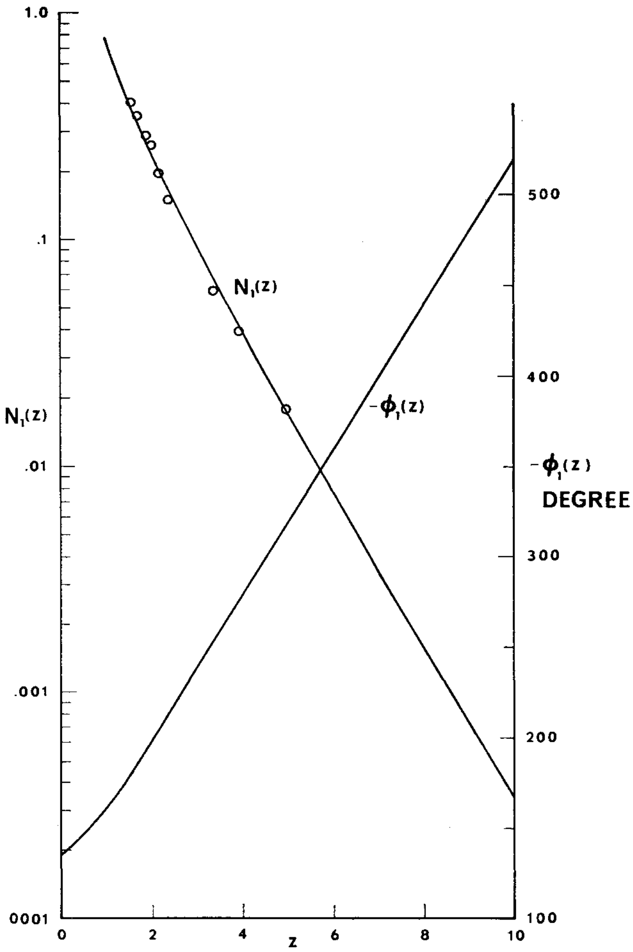

Fig. 1 The functions N_1 (z) and ϕ_1 (z).

dissipated into heat per half-cycle of motion is

$$W_0 = -\int_0^{\pi/\omega} \tilde{F}\cdot\tilde{V}_c dt = 2\pi a V_0 \int_0^{\pi/\omega} \tau_w \cos(\beta - \gamma)\cos\omega t dt \quad (8)$$

Exact and Approximate Solutions to the Equations

The solutions to Eqs. (2) and (3) can be obtained by the separation of variables technique. The resulting ordinary differential equations obtained for the spatial variable can be expressed in terms of modified Bessel functions $K_0(z)$ and $K_1(z)$ with complex arguments. Specifically, the solution of Eq. (2) is written

$$v(r,t) = \frac{K_1[(i\omega/\nu)^{1/2}r]}{K_1[(i\omega/\nu)^{1/2}a]} V_0 \cos\beta e^{i\omega t} \quad (9)$$

By using the relationship

$$K_1[(i\omega/\nu)^{1/2}r] = iN_1[(\omega/\nu)^{1/2}r]\exp\{i\phi_1[(\omega/\nu)^{1/2}r]\} \quad (10)$$

one obtains

$$v(r,t) = \frac{N_1[(\omega/\nu)^{1/2}r]}{N_1[(\omega/\nu)^{1/2}a]} V_0 \cos\beta \exp\left(i\left\{\omega t + \phi_1\left[\left(\frac{\omega}{\nu}\right)^{1/2} r\right] - \phi_1\left[\left(\frac{\omega}{\nu}\right)^{1/2} a\right]\right\}\right) \quad (11)$$

where the functions $N_1(z)$ and $\phi_1(z)$, tabulated in Ref. 3 (p. 433), are plotted in Fig. 1.

The solution of Eq. (3) was first obtained by Stokes[4] in 1886 and is written

$$w(r,t) = \frac{K_0[(i\omega/\nu)^{1/2}r]}{K_0[(i\omega/\nu)^{1/2}a]} V_0 \sin\beta e^{i\omega t} \quad (12)$$

Again, using the relationship

$$K_0[(i\omega/\nu)^{1/2}r] = N_0[(\omega/\nu)^{1/2}r]\exp\{i\phi_0[(\omega/\nu)^{1/2}r]\} \quad (13)$$

one obtains

$$w(r,t) = \frac{N_0[(\omega/\nu)^{1/2}r]}{N_0[(\omega/\nu)^{1/2}a]} V_0 \sin\beta \exp\left(i\left\{\omega t + \phi_0\left[\left(\frac{\omega}{\nu}\right)^{1/2} r\right] - \phi_0\left[\left(\frac{\omega}{\nu}\right)^{1/2} a\right]\right\}\right) \quad (14)$$

where the functions $N_0(z)$ and $\phi_0(z)$ are listed in Ref. 3 (p. 433) and are shown in Fig. 2. The velocity profiles given by Eqs. (11) and (14) for $v(r,t)$ and $w(r,t)$ are shown in Figs. 3 and 4 for various intervals of the cycle and $\alpha = \frac{1}{2}$. Since a large value of α was chosen for these profiles, the asymptotical expansions [Eqs. (19–22)] allow Eqs. (11) and (14) to be approximately the same and hence Figs. 3 and 4 are remarkably similar. By using the differential formulas (Ref. 3, p. 376),

$$(d/dz)[K_0(\alpha z)] = -\alpha K_1(\alpha z) \quad (15)$$

$$(d/dz)[K_1(\alpha z)] = -\alpha K_0(\alpha z) - (1/z)K_1(\alpha z) \quad (16)$$

and combining these relationships with Eqs. (6, 7, 9, 10, 12, and 13), the values of the shear stresses become

$$\tau_{r\theta} = \mu \frac{\alpha}{a}\frac{N_0(\alpha)}{N_1(\alpha)} LV_0 \cos\beta \cos(\omega t + \delta) \quad (17)$$

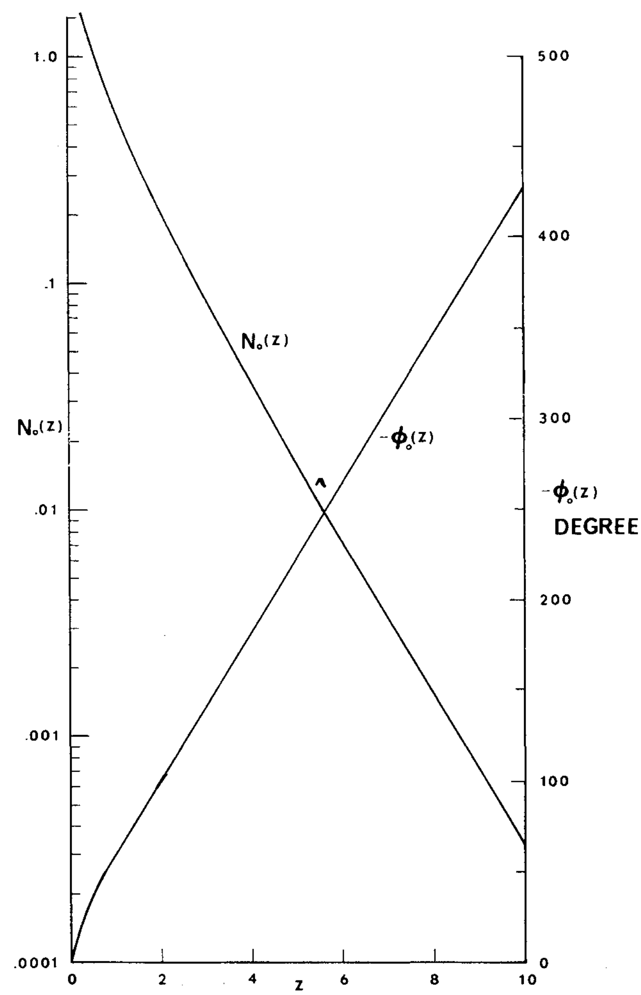

Fig. 2 The functions N_0 (z) and ϕ_0 (z).

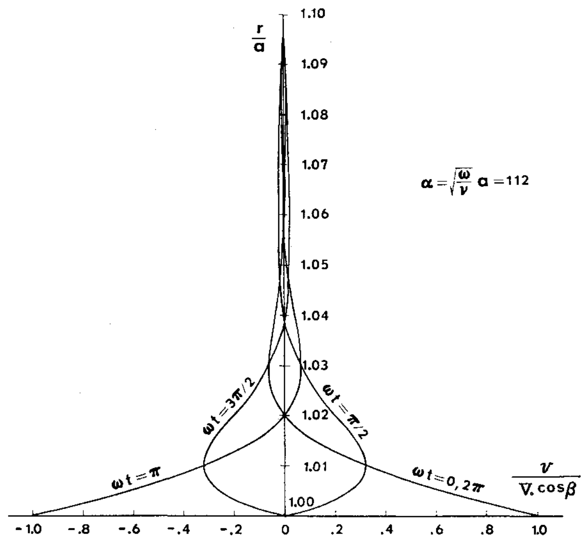

Fig. 3 $v/(V_0 \cos\beta)$ **for** $\alpha = (\omega/\nu)^{\frac{1}{2}}\, a = 112$.

$$\tau_{rz} = \mu\, \frac{\alpha}{a}\, \frac{N_1(\alpha)}{N_0(\alpha)}\, V_0 \sin\beta \cos(\omega t + \zeta) \tag{18}$$

where

$$\alpha = (\omega/\nu)^{1/2}a,\ \zeta = \phi_1(\alpha) - \phi_0(\alpha) - \pi/4$$
$$\eta = \phi_0(\alpha) - \phi_1(\alpha) + 3\pi/4$$
$$\tan\delta = \frac{\sin\eta}{\cos\eta - (2/\alpha)N_1(\alpha)/N_0(\alpha)}$$

and

$$L = \left[\left(\cos\eta - \frac{2}{\alpha}\cdot\frac{N_1(\alpha)}{N_0(\alpha)}\right)^2 + \sin^2\eta\right]^{1/2}$$

where the functions $N_0(z)$ and $\phi_0(z)$ are listed in Ref. 3.

The critical parameter in these results is $(\omega/\nu)^{1/2}a$. For some practical oceanographic problems, the parameters can be bounded by

$$1.0 \times 10^{-5} \leq \nu \leq 2.0 \times 10^{-5}\ \text{ft}^2/\text{sec}$$
$$1 \leq \omega/2\pi \leq 10\ \text{cps}$$

thus

$$560 \leq (\omega/\nu)^{1/2} \leq 1770\ \text{ft}^{-1}$$

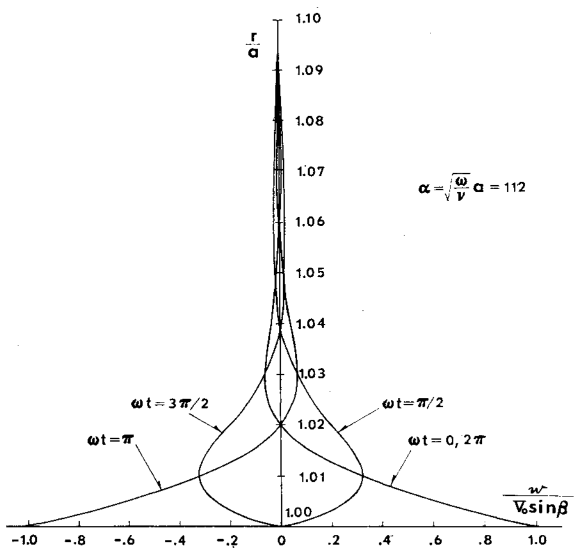

Fig. 4 $w/(V_0 \sin\beta)$ **for** $\alpha = (\omega/\nu)^{\frac{1}{2}}\, a = 112$.

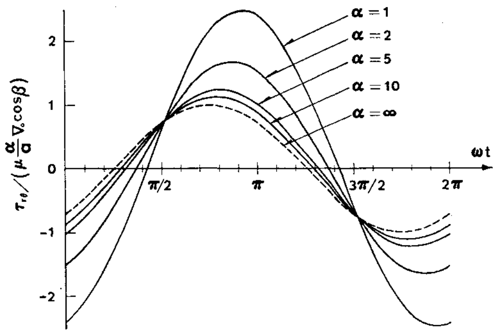

Fig. 5 The dimensionless shear stresses $\tau_{r\theta}/[\mu(\alpha/a)$-$V_0\cos\beta]$.

The minimum value of $(\omega/\nu)^{1/2}a$ for a $\frac{1}{4}$-in.-diam rod might be approximately 5.05.

The results can be greatly simplified if $(\omega/\nu)^{1/2}a \equiv \alpha \gg 1$ by using the asymptotic formulas Ref. 3, p. 383

$$N_0(z) \simeq \left(\frac{\pi}{2z}\right)^{1/2} \exp(-z/2^{1/2}) \times \left\{1 - \frac{1}{8(2)^{1/2}z} + \frac{1}{256z^2} + \frac{133}{2048(2)^{1/2}z^3}\right\} \tag{19}$$

$$\phi_0(z) \simeq -\frac{z}{2^{1/2}} - \frac{1}{8}\pi + \frac{1}{8(2)^{1/2}z} - \frac{1}{16z^2} + \frac{25}{384(2)^{1/2}z^3}\ \text{rad} \tag{20}$$

$$N_1(z) \simeq \left(\frac{\pi}{2z}\right)^{1/2} \exp(-z/2^{1/2}) \times \left\{1 + \frac{3}{8(2)^{1/2}z} + \frac{9}{256z^2} - \frac{327}{2048(2)^{1/2}z^3}\right\} \tag{21}$$

$$\phi_1(z) \simeq -\frac{z}{2^{1/2}} - \frac{5}{8}\pi - \frac{3}{8(2)^{1/2}z} + \frac{3}{16z^2} - \frac{21}{128(2)^{1/2}z^3}\ \text{rad} \tag{22}$$

If one neglects terms of order of $1/z$ in Eqs. (19–22) and substitutes these simplified expressions into Eqs. (17) and (18), the latter equations become

$$\tau_{r\theta} \simeq \mu(\alpha/a)V_0 \cos\beta \cos(\omega t - 3\pi/4) \tag{23}$$

and

$$\tau_{rz} \simeq \mu(\alpha/a)V_0 \sin\beta \cos(\omega t - 3\pi/4) \tag{24}$$

As a result of these approximations, the simplified solutions given by Eqs. (23) and (24) are independent of the diameter of the rod.

A comparison of the approximate and exact expressions for $\tau_{r\theta}$ and τ_{rz} over one cycle with $\alpha \equiv (\omega/\nu)^{1/2}a$ as a parameter are shown in Figs. 5 and 6.

Comparison and Application of the Results

The exact velocity profiles for $v(r,t)$ and $w(r,t)$ are given by Eqs. (11) and (14). The maximum particle velocity for

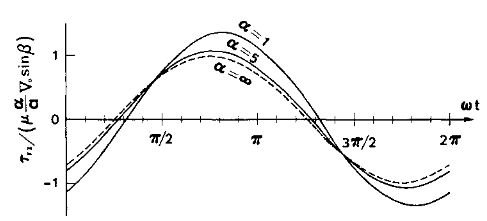

Fig. 6 The dimensionless shear stresses $\tau_{rz}/[\mu(\alpha/a)$-$V_0\sin\beta]$.

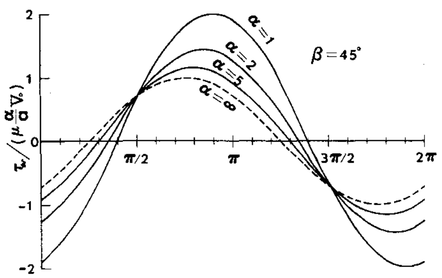

Fig. 7 The dimensionless total shear stresses $\tau w/[\mu(\alpha/a)V_0]$ for $\beta = 45°$.

the pure rotatory oscillation ($\beta = 0°$) is given by

$$(v/V_0)_{\max} = N_1[(\omega/\nu)^{1/2}r]/N_1[(\omega/\nu)^{1/2}a] \tag{25}$$

and this analytical result is in excellent agreement with the data obtained in an experiment performed in oil with $\alpha \simeq 1.33$ by Winny.[5] This data is superimposed in Fig. 1.

For the special case of $\alpha \gg 1$, the magnitude of the shear stress becomes

$$\tau_w = \mu(\omega/\nu)^{1/2}\, V_0 \cos(\omega t - 3\pi/4) \text{(oscillating cylinder)} \tag{26}$$

and is independent of the angle β and the radius of the rod. The drag force is in the opposite direction to the velocity since

$$\tan\gamma = \tau_{rz}/\tau_{r\theta} \simeq \tan\beta \tag{27}$$

It is interesting to compare the expression for the shear stress on the wall of an oscillating flat plate (Ref. 6);

$$\tau_0 = \mu(\omega/\nu)^{1/2} V_0 \cos(\omega t - 3\pi/4) \text{(oscillating flat plate)} \tag{28}$$

which is identical to the simplified expression given previously.

For the case α of the order 1, the results for τ_w will be quite different as shown in Fig. 7, which compare the exact and approximate values of τ_w over the range of values of α and $\beta = 45°$.

It should also be noted that for small values of the parameter the direction of the drag force (angle γ) varies over the cycle and is not equal to the angle β, the direction of the oscillatory motion.

An expression for the drag coefficient can be obtained by equating the work done on the fluid by a hypothesized drag force of the form

$$\tilde{F}_H = -CV_c{}^n\tilde{e} \tag{29}$$

which is

$$\int_0^{2\pi} \tilde{F}_H \cdot \tilde{V}_c dt = C \int_0^{\pi/\omega} V_c{}^{n+1} dt \tag{30}$$

to the exact expression given by Eq. (8). This gives

$$C = \frac{2\pi a}{V_0{}^n}\left(\int_0^{\pi/\omega} \cos(\gamma - \beta)\tau_w \cos\omega t dt \Big/ \int_0^{\pi/\omega} \cos^{n+1} \omega t dt\right) \tag{31}$$

where $V_c = V_0 \cos\omega t$.

The only meaningful case for which the drag coefficient C is independent of V_0 is $n = 1$ since τ_w is proportional to V_0.

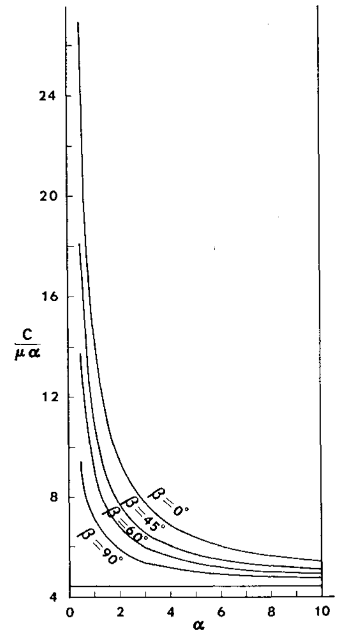

Fig. 8 Variation of $C/(\mu\alpha)$ with α.

By using the approximate expression given by Eq. (26) for τ_w, one obtains from Eq. (33) the simple expression for the case $n = 1$ of

$$C \approx 4.45\mu\alpha \tag{32}$$

Using the values of α of 100 and $\mu = 3 \times 10^{-5}$ lb-sec/ft^2, the drag coefficient is $C \approx 0.0134$. For the case of the tangential drag coefficient on a cable with uniform flow, Pode[7] has experimentally obtained

$$C_F = F/\tfrac{1}{2}\rho V_0{}^2 d \approx 0.02$$

A comparison of the exact values of C for $\beta = 0°$ and $\beta = 90°$ given by Eq. (31) and the approximate expression Eq. (32) are shown in Fig. 8 for various values of α.

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